Actual computation for the complexified hyperbolic volume conjecture (Volume Conjecture and Its Related Topics)

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Actual computation for the complexified hyperbolic volume conjecture

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Absract. In this report, some computations to check the complexified hyperbolic volume conjecture in [4] are explained. This conjecture concerns the first term of the asymptotics of $\lim_{N\to\infty} J_N(K)$ of a hyperbolic knot K. In the workshop at IIAS, Hikami explained his observation about the second term of the asymptotics of $\lim_{N\to\infty} J_N(K)$, which is also positively cheched in the computation of this report.

Acknowlagement. I would like to thank all participants of the workshop at IIAS for useful discussion, especially K. Hikami for introducing his technique to do actual computation by using Pari-Gp.

1. Introduction.

1.1. Complexified hyperbolic volume conjecture. For a hyperbolic knot K in S^3, let $\mathrm{Vol}(K)$ and $\mathrm{CS}(K)$ be the hyperbolic volume and the Chern-Simons invariant respectively of the complement of K. Then the *complexified hyperbolic volume conjecture* [4] is the following formula to explain $\mathrm{Vol}(K)$ and $\mathrm{CS}(K)$ as a certain limit of the colored Jones invariants.

(1) $\boxed{\textbf{Complexified hyperbolic volume conjecture}}$ $$\lim_{N\to\infty} J_N(K) = \exp\left(\frac{N}{2\pi}\left(\mathrm{Vol}(K)+\sqrt{-1}\ \mathrm{CS}(K)\right)\right)$$

Here $J_N(K)$ is the colored Jones polynomial corresponding to the N dimensional representation $\mathcal{U}_q(sl_2)$ with the parameter q specialized to $\exp(2\pi\sqrt{-1}/N)$, the primitive N-th root of unity. This invariant $J_N(K)$ is proved in [3] to be equal to the Kashaev's invariant $K_N(K)$. The exact meaning of the imaginary part is given in (3).

This congecture is based on the following Kashaev conjecture. In [1], Kashaev conjectured that

(2) $\boxed{\textbf{Kashaev conjecture}}$ $$\lim_{N\to\infty} |K_N(K)| = \exp \frac{N\,\mathrm{Vol}(K)}{2\pi}.$$

He checked this relation exactly for the figure-eight knot 4_1 and numerically for the knots 5_2 and 6_1.

The arguments of $K_N(K)$ $(= J_N(K))$ are investigated in [4] and the complexified hyperbolic volume conjecture (1) is proposed. To give the exact meaning of the imaginary part of (1), it may be better to consider the following relation:

$$\lim_{N\to\infty} \frac{J_{N+1}(K)}{J_N(K)} = \exp\left(\frac{1}{2\pi}\left(\mathrm{Vol}(K) + \sqrt{-1}\ \mathrm{CS}(K)\right)\right), \tag{3}$$

which is cheched numerically for some examples in the rest of this report.

Kashaev conjectured (2) for hyperbolic knots, and it is generalized in [3] for any knot K as follows:

(4) $\boxed{\text{Volume conjecture}}$ $$|J_N(K)| \underset{N\to\infty}{\sim} \exp\left(N\,\frac{v_3\,|S^3\setminus K|}{2\,\pi}\right),$$

where $|S^3 \setminus K|$ denotes Gromov's simplicial volume of $S^3 \setminus K$, and v_3 is the volume of the ideal regular tetrahedron of the hyperbolic 3-space H^3, i.e.

$$v_3 = 1.01494160640965362502120255 4$$

It may be natural to consider about a complexification of the volume conjecture, which might be the following form.

(5)

$\boxed{\text{Complexified volume conjedture}}$ $$J_N(K) \underset{N\to\infty}{\sim} \exp\left(N\left(\frac{v_3\,|S^3\setminus K|}{2\,\pi} + \sqrt{-1}\ \mathrm{CS}(K)\right)\right),$$

or, its quotient version

$$\frac{J_{N+1}(K)}{J_N(K)} \underset{N\to\infty}{\sim} \exp\left(N\left(\frac{v_3\,|S^3\setminus K|}{2\,\pi} + \sqrt{-1}\ \mathrm{CS}(K)\right)\right). \tag{6}$$

2. Hikami's observation.

Hikami observed that

$$(7) \quad \boxed{\text{Hikami's observation}} \qquad 2\pi \log |J_N(K)| \underset{N\to\infty}{\sim} \mathrm{Vol}(K)\, N + 3\pi \log N + O\left(\frac{1}{N}\right)$$

for several hyperbolic prime knots. The volume $\mathrm{Vol}(K)$ in the first term corresponds to the Kashaev's conjecture (2). The second term explains a misterious behavior of $|J_N(K)|$, since the coefficient 3π appears for every prime knot he checked. Moreover, Kashaev and Tirkkonen [2] proved that

$$(8) \qquad |J_N(K)| \sim N^{3/2}$$

for any torus knot K. This implies that

$$(9) \qquad 2\pi \log |J_N(K)| \sim 3\pi \log N.$$

Now reformulate (7) for $J_{N+1}(K)/J_N(K)$ to compare the complexified hyperbolic volume conjecture. Since

$$\log(N+1) - \log N \underset{N\to\infty}{\sim} \frac{1}{N},$$

Hikami's observation (7) is reformulated as follows.

$$(10) \qquad 2\pi \log \left| \frac{J_{N+1}(K)}{J_N(K)} \right| \underset{N\to\infty}{\sim} \mathrm{Vol}(K) + \frac{3\pi}{N} + O\left(\frac{1}{N^2}\right).$$

This relation seems to be true for all the examples given in this report.

3. Actual computations for several knots

3.1. Preliminaries. Let N be a positive integer and

$$q = \exp 2\pi\sqrt{-1}/N.$$

Let

$$(x)_k = \prod_{i=1}^{k}(1 - x^i).$$

It is known that

$$(q)_{N-1} = (\bar{q})_{N-1} = N,$$

and so

$$\frac{1}{(q)_i} = \frac{(\bar{q})_{N-1-i}}{N}, \qquad \frac{1}{(\bar{q})_i} = \frac{(q)_{N-1-i}}{N}. \tag{11}$$

3.2. Figure-eight knot 4_1. For the figure-eight knot K,

$$J_N(K) = \sum_{i=0}^{N-1}(q)_i(\bar{q})_i \tag{12}$$

Since $(q)_i(\bar{q})_i$ is a positive real number, numerical computation of the summension may have good accuracy. By using "pari-Gp 2.0.14" [5], which uses 28 digits for real numbers, $J_{N+1}/J_N(K)$ is computed by the following program.

Program. The feature of this program is to compute the formula consisting of a sum of the terms of $(q)_i$ and $(\bar{q})_i$ as a polynomial modulo $x^N - 1$. We replace q by an indeterminate x and compute everything as a polynomial in x modulo $x^N - 1$. Here we use the function of Pari to handle polynomials modulo a palynomial. At the end of the computation of $J_N(K)$, x is replaced by $q = \exp 2\pi\sqrt{-1}/N$.

The following program is for $N = 40$ case. The parameter `l` represents a list for

$$(x)_i \mod x^N - 1.$$

The i-th component of `l` is $(x)_{i-1}$. Similarly, the papareter `lm` is a list for

$$(x^{-1})_i \mod x^N - 1,$$

and `labs` for

$$(x)_{i-1}(x^{-1})_i \mod x^N - 1.$$

The parameters **ansn1** and **ansn2** contain the value of $J_N(K)$ and J_{N+1} respectively. In Pari, a pollynomial $P(x)$ modulo a polynomial $Q(x)$ is represented by

```
Mod(P(x), Q(x))
```

and the part `P(x)` is obtained by

```
component(Mod(P(x), Q(x)), 2)
```

By using the above conventions, the program to compute $J_{N+1}(K)/J_N(K)$ is the following.

```
N = 40
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
lm = listcreate(N)
listinsert(lm, 1, 1)
for(i=2, N, listinsert(lm, Mod(1-x^(N-i+1), x^N-1)*lm[i-1], i))
labs = listcreate(N)
for(i=1, N, listinsert(labs, l[i]*lm[i], i))
ans = sum(i=0, N-1, labs[i+1])
ansn1 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
N = N+1
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
lm = listcreate(N)
listinsert(lm, 1, 1)
for(i=2, N, listinsert(lm, Mod(1-x^(N-i+1), x^N-1)*lm[i-1], i))
labs = listcreate(N)
for(i=1, N, listinsert(labs, l[i]*lm[i], i))
ans = sum(i=0, N-1, labs[i+1])
ansn2 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
2*Pi*log(ansn2/ansn1)
```

Results. The results of $\dfrac{J_{N+1}(K)}{J_N(K)}$ are obtained as in Table 1.

N	$2\pi \log (J_{N+1}(K)/J_N(K))$
2	6.003655472238323292231589540
3	4.592301654877167312392618658
4	3.930663682692356853738064934
5	3.563916633541326127285769653
6	3.333164709424714865870562762
7	3.172317947938872936912527868
8	3.051144721847726154953374741
9	2.954782008769767302140299724
10	2.875358671602468461692196949
11	2.808342526942869241816837690
12	2.750885889902206110562749497
13	2.701051329730451549828435963
15	2.618970065418565923553326505
17	2.554332587366309720234150867
20	2.479853832424784111071256298
30	2.334775258885131229477377674
40	2.260321485477360972294568843
50	2.215072743903543628382313096
60	2.184671081393873149842712520
70	2.162841133980227344819650089
80	2.146406409346649171123325140
90	2.133587081931110608322566384
100	2.123308518493499319808821058
110	2.114883540456906000598732813
120	2.107852260578843496981192338
130	2.101895294219505133603807249
140	2.096783913363572891972780032
150	2.092350013953243978620902884
160	2.088467279743770433433590337
180	2.081989688634303299075240685
200	2.076801889279751010583329074
220	2.072553542846379820983370778
250	2.067451035444170566134831307
280	2.063438486960197234222353449
310	2.060200359742050080769531549
350	2.056744192244877957986999400
380	2.054628461780599867892655099
410	2.052821688871750122645027257
440	2.051260803848047983628309857
470	2.049898809746513360171439063
500	2.048699969018691584059221985
600	2.045568401020993284743979175
700	2.043330449942604266902872242
1000	2.039299793540753767753924093
$2\pi \log \frac{J_{1001}(K)}{J_{1000}} - \frac{3\pi}{1000}$	2.029875015579984388038536163
$\mathrm{Vol}(K)$	2.029883212819307250042405109

TABLE 1

Graph. The points $\left(\frac{1}{N}, \frac{J_{N+1}(K)}{J_N(K)}\right)$ of the above data is plotted as follows.

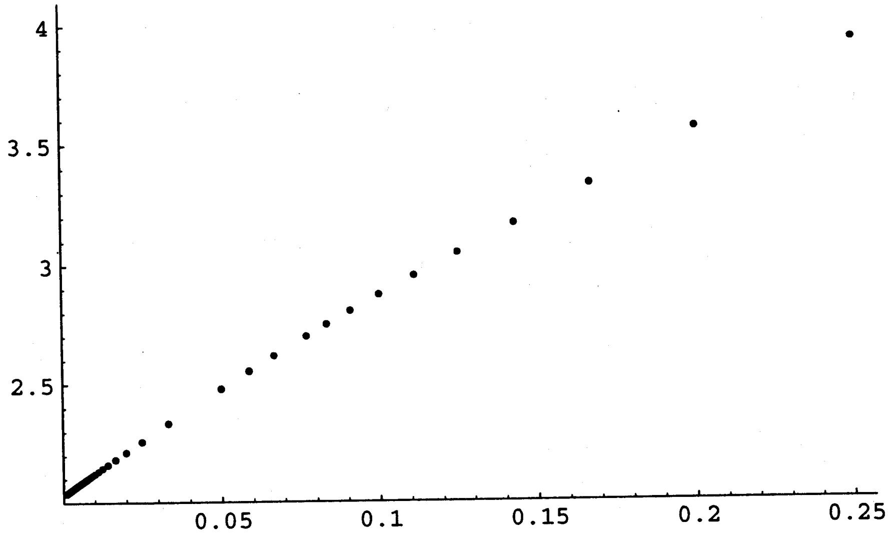


FIGURE 1. Plotting of the points $\left(\frac{1}{N}, 2\pi \log \frac{J_{N+1}(K)}{J_N(K)}\right)$ of the knot 4_1.

Fitting. From the above result, we can predict the actual limit by estimating the asymptotics of $J_N(K)$ by fitting with certain function, which is determined by the least square method. Here, we try to use the function of the form

$$a_0 + \frac{a_1}{N} + \frac{a_2}{N^2}. \tag{13}$$

a_0, a_1 and a_2 are obtained by the following function of Matehmatica

```
Fit[l /. {x_, y_} -> {1/x, y}, {1, x, x^2}, x]
```

where `l` is a list of the pairs

$$\{N, J_{N+1}(K)/J_N(K)\}$$

in the above table with $N \geq 40$. The result of the fitting is

$$2.02988 + 9.42629 \frac{1}{N} - 8.34016 \frac{1}{N^2}.$$

Note that the constant term is equal to the volume of $S^3 \setminus K$ up to 6 digits, and the coefficient of `x` is almost equal to $3\pi = 9.42478\ldots$.

3.3. **Knot 5_2.** Let K be the knot 5_2. Then

$$J_N(K) = \sum_{i=1}^{N-1} \sum_{j=1}^{i} \frac{(q)_i^{\,2}}{(\bar{q})_j}.$$

This knot is achiral and $J_N(K)$ has a non-trivial imaginary part. The following results suggest that

$$\lim_{N\to\infty} \log \frac{J_{N+1}(K)}{J_N(K)} = \mathrm{Vol}(K) + \sqrt{-1}\ \mathrm{CS}(K),$$

where

$$\mathrm{Vol}(K) = 2.82812208833, \qquad \mathrm{CS}(K) = -3.02412837657.$$

The program to compute $J_{N+1}(K)/J_N(K)$ for $N = 40$ is

```
N = 40
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
l2 = listcreate(N)
for(i=1, N, listinsert(l2, l[i]*l[i], i))
ans = sum(i=0, N-1, l2[i+1]*sum(j=0,i, \
  l[N-1-j+1]*Mod(x^component(Mod(-j*(i+1),N), 2), x^N-1)))
ansn1 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
N = N+1
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
l2 = listcreate(N)
for(i=1, N, listinsert(l2, l[i]*l[i], i))
ans = sum(i=1, N-2, l2[i+1]*sum(j=0,i, \
  l[N-1-j+1]*Mod(x^component(Mod(-j*(i+1),N), 2), x^N-1)))
ansn2 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
2*Pi*log(ansn2*(N-1)/ansn1/N)
```

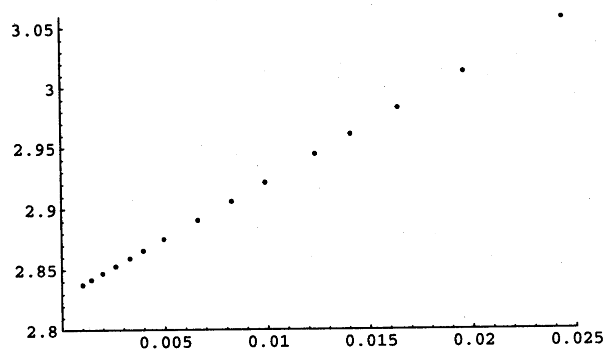
At the last line, `N` and `N-1` are added since, in the computation of `ans`, we use the relation (11).

The results are given in Table 2.

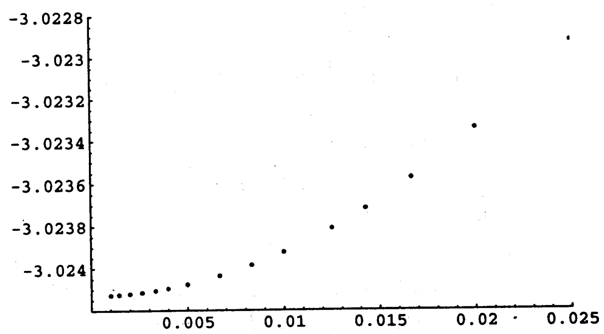
N	$2\,\pi\ \log \dfrac{J_{N+1}(K)}{J_N(K)}$
40	3.058223721261842722613885956 - 3.022924613281720287391974968 $\sqrt{-1}$
50	3.013081508530188353573854822 - 3.023340368517507069134855780 $\sqrt{-1}$
60	2.982744318753580696821772299 - 3.023574042878935429645720640 $\sqrt{-1}$
70	2.960955404961739170749114151 - 3.023717381786374852930574631 $\sqrt{-1}$
80	2.944548269170450112446966301 - 3.023811574968472287718611711 $\sqrt{-1}$
100	2.921483906108228993018469212 - 3.023923719027833555669502480 $\sqrt{-1}$
120	2.906046421388666000282542398 - 3.023985374930307234443986632 $\sqrt{-1}$
150	2.890559881907537128372001511 - 3.024036295143969179028770901 $\sqrt{-1}$
200	2.875024234226941620327156350 - 3.024076266558545340852410631 $\sqrt{-1}$
250	2.865679250969538531562099056 - 3.024094905811349375139149331 $\sqrt{-1}$
300	2.859439423619654229923900269 - 3.024105077353483138303449159 $\sqrt{-1}$
380	2.852862676601409465863918924 - 3.024113818437089706026655831 $\sqrt{-1}$
500	2.846936140234120797452382677 - 3.024119948850536105286710779 $\sqrt{-1}$
$2\pi \log \frac{J_{501}(K)}{J_{500}(K)} - \frac{3\pi}{500}$	2.828086584312582038021606817 - 3.024119948850536105286710779 $\sqrt{-1}$
$\mathrm{Vol}(K)+\sqrt{-1}\,\mathrm{CS}(K)$	2.82812208833 -3.02412837657 $\sqrt{-1}$

TABLE 2. $\mathrm{CS}(K) = 2\pi^2\,\mathrm{cs}(K)$ where $\mathrm{cs}(K)$ is the Chern-Simons inariant obtained by SnapPea.

Graphs. The real and imaginary parts of the points $\left(\dfrac{1}{N}, 2\pi\ \log \dfrac{J_{N+1}(K)}{J_N(K)}\right)$ are plotted as follows.



real part



imaginary part

FIGURE 2. Plotting of the points $\left(\dfrac{1}{N}, 2\pi\ \log \dfrac{J_{N+1}(K)}{J_N(K)}\right)$ of the knot 5_2.

Fitting. The result of the fitting is

$$2.82812 - 3.02413\sqrt{-1} + \left(9.42398 + 0.00306762\sqrt{-1}\right)\frac{1}{N} - \left(8.80117 - 1.82018\sqrt{-1}\right)\frac{1}{N^2}.$$

3.4. **Knot 6_1.** Let K be the knot 6_1. Then

$$J_N(K) = \sum_{\substack{0 \le m \le N-1 \\ 0 \le k+l \le m}} \frac{|(q)_m|^2}{(\bar{q})_k\,(q)_l}\, q^{(m-k-ll)(m-k+1)}.$$

The program to compute $J_{N+1}(K)/J_N(K)$ for $N = 40$ is

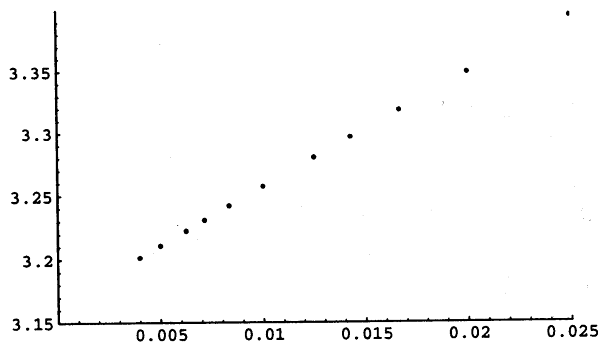
```
N = 40
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
lm = listcreate(N)
listinsert(lm, 1, 1)
for(i=2, N, listinsert(lm, Mod(1-x^(N-i+1), x^N-1)*lm[i-1], i))
labs = listcreate(N)
for(i=1, N, listinsert(labs, l[i]*lm[i], i))
ans = sum(m=0, N-1, labs[m+1]*\
  sum(k=0, m, lm[N-k-1+1]*\
  sum(ll=0, m-k, l[N-ll-1+1]*\
  Mod(x^component(Mod((m-k-ll)*(m-k+1), N), 2), x^N-1)\
  )))
ans1 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
N = N+1
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
lm = listcreate(N)
listinsert(lm, 1, 1)
for(i=2, N, listinsert(lm, Mod(1-x^(N-i+1), x^N-1)*lm[i-1], i))
labs = listcreate(N)
for(i=1, N, listinsert(labs, l[i]*lm[i], i))
ans = sum(m=0, N-1, labs[m+1]*\
  sum(k=0, m, lm[N-k-1+1]*\
  sum(ll=0, m-k, l[N-ll-1+1]*\
  Mod(x^component(Mod((m-k-ll)*(m-k+1), N), 2), x^N-1)\
  )))
ans2 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
2*Pi*log(ans2*(N-1)^2/ans1/N^2)
```

The results are given in Table 3

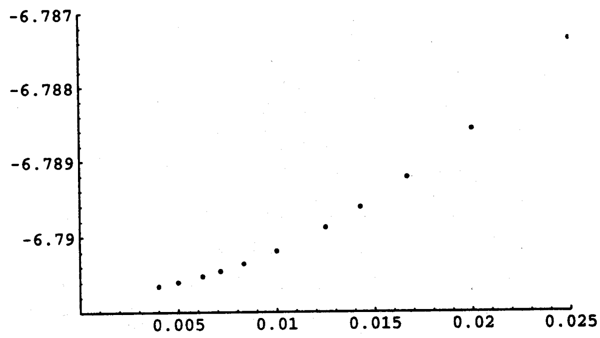
N	$2\pi\ \log \dfrac{J_{N+1}(K)}{J_N(K)}$
40	3.394414401189434606382674303 - 6.787357383517771950431779748$\sqrt{-1}$
50	3.349042577638057792369311322 - 6.788573830549551769685488316$\sqrt{-1}$
60	3.318663899636576334652428537 - 6.789220588929412990320766423$\sqrt{-1}$
70	3.296853139714908393514056469 - 6.789619052513418482514938102$\sqrt{-1}$
80	3.280431935209578682617705935 - 6.789879359008531621725741068$\sqrt{-1}$
100	3.257351525805095407490069456 - 6.790187222588042970544664690$\sqrt{-1}$
120	3.241905609350914165102503653 - 6.790355420351823054866703734$\sqrt{-1}$
140	3.230843837118369372290452265 - 6.790457237072288734026314403$\sqrt{-1}$
160	3.222531628510898663690021823 - 6.790523508867061899847731603$\sqrt{-1}$
200	3.210871626400388744216603973 - 6.790601669404146304505355465$\sqrt{-1}$
250	3.201524345448275380003116810 - 6.790651846104617907062217087$\sqrt{-1}$
$2\pi \log \frac{J_{251}(K)}{J_{250}(K)} - \frac{3\pi}{250}$	3.163825233605197861141565089 - 6.790651846104617907062217087$\sqrt{-1}$
$\mathrm{Vol}(K)+$ $\sqrt{-1}\,\mathrm{CS}(K)$	3.1639632289 -6.7907414993 $\sqrt{-1}$

TABLE 3. $\mathrm{CS}(K) = -2\pi^2 \operatorname{cs}(K) + \pi^2$.

Graphs. The real and imaginary parts of the points $\left(\dfrac{1}{N}, \dfrac{J_{N+1}(K)}{J_N(K)}\right)$ are plotted as follows.



real part



imaginary part

FIGURE 3. Plotting of the points $\left(\dfrac{1}{N}, \dfrac{J_{N+1}(K)}{J_N(K)}\right)$ of the knot 6_1.

Fitting. The result of the fitting is

$$(3.16404 - 6.79075\sqrt{-1}) + (9.40652 + 0.00370946\sqrt{-1})\ \frac{1}{N} - (7.70212 - 5.27915\sqrt{-1})$$

3.5. **Knot 6_3**. Let K be the knot 6_3. Then

$$J_N(K) = \sum_{\substack{k,l,m \geq 0 \\ k+l+m \leq N-1}} \left| \frac{(q)_{k+l+m}}{(\bar{q})_l\,(q)_m} \right|^2 (q)_{k+l}\,(\bar{q})_{m+k}\,q^{(m-l)(k+1)}.$$

The program to compute $J_{N+1}(K)/J_N(K)$ for $N = 40$ is

```
N = 40
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
lm = listcreate(N)
listinsert(lm, 1, 1)
for(i=2, N, listinsert(lm, Mod(1-x^(N-i+1), x^N-1)*lm[i-1], i))
labs = listcreate(N)
for(i=1, N, listinsert(labs, l[i]*lm[i], i))
ans = sum(m=0, N-1, labs[N-1-m+1]*\
  sum(p=0, N-1-m, labs[p+m+1]*lm[p+1]*\
  sum(k=0, p, labs[N-1-p+k+1]*l[m+k+1]*\
  Mod(x^component(Mod(-(m-p+k)*(k+1), N), 2), x^N-1))))
ans1 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
N = N+1
l = listcreate(N)
listinsert(l, 1, 1)
for(i=2, N, listinsert(l, Mod(1-x^(i-1), x^N-1)*l[i-1], i))
lm = listcreate(N)
listinsert(lm, 1, 1)
for(i=2, N, listinsert(lm, Mod(1-x^(N-i+1), x^N-1)*lm[i-1], i))
labs = listcreate(N)
for(i=1, N, listinsert(labs, l[i]*lm[i], i))
ans = sum(m=0, N-1, labs[N-1-m+1]*\
  sum(p=0, N-1-m, labs[p+m+1]*lm[p+1]*\
  sum(k=0, p, labs[N-1-p+k+1]*l[m+k+1]*\
  Mod(x^component(Mod(-(m-p+k)*(k+1), N), 2), x^N-1))))
ans2 = subst(component(ans, 2), x, exp(2*Pi*sqrt(-1)/N))
2*Pi*log(ans2*(N-1)^4/ans1/N^4)
```

The results are given in Table 4.

N	$2\pi \log \dfrac{J_{N+1}(K)}{J_N(K)}$
30	5.991757632388930862295686837
40	5.920010510909063767712690688
47	5.887310870362322038241138727
50	5.876009180047075973936088402
60	5.846282921844738453303387249
70	5.824859282414985211663083205
80	5.808687819822659085249294793
94	5.791733883431946311125566885
100	5.785898993155213353224223121
120	5.770610335748061213979602476
150	5.755245033266310556638346366
$2\pi \log \frac{J_{251}(K)}{J_{250}(K)} - \frac{3\pi}{150}$	5.692413180194514691869093498
$\mathrm{Vol}(K)$	5.69302109128

TABLE 4

Graph. The real and imaginary parts of the points $\left(\dfrac{1}{N}, 2\pi \log \dfrac{J_{N+1}(K)}{J_N(K)}\right)$ are plotted as follows.

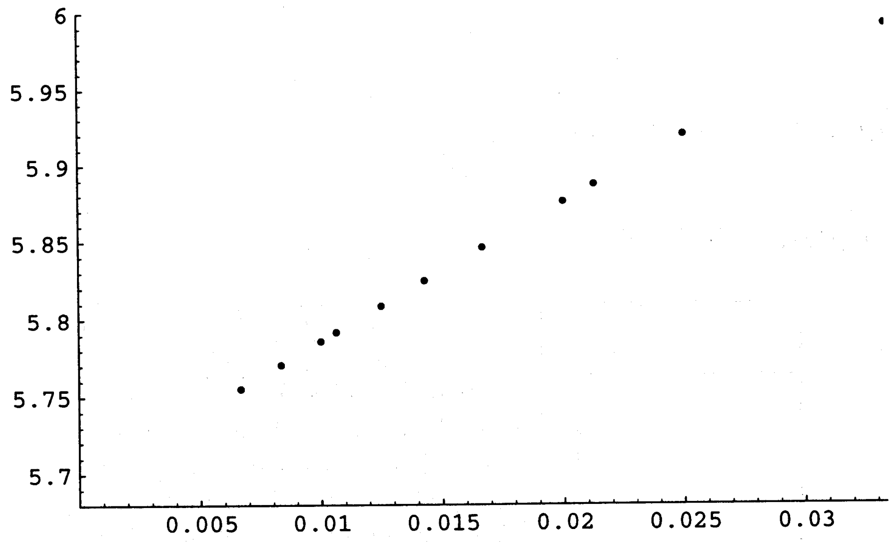


FIGURE 4. Plotting of the points $\left(\dfrac{1}{N}, 2\pi \log \dfrac{J_{N+1}(K)}{J_N(K)}\right)$ of the knot 6_3.

Fitting. The result of the fitting is

$$5.69297 + 9.43385 \frac{1}{N} - 14.1061 \frac{1}{N^2}.$$

3.6. **Knot** 8_9. Let K be the knot 8_9. Then

$$J_N(K) = \sum_{\substack{0 \le l, m_1, m_2, n_1, n_2 \le N-1 \\ m_1+n_1, m_2+n_2 \le l \\ m_1+m_2 \le l}} \left| \frac{(q)_{l-m_1} (q)_l (q)_{l-m_2}}{(q)_{m_1} (q)_{m_2} (q)_{n_1} (q)_{n_2}} \right|^2 \frac{(\bar{q})_{l-n_1} (q)_{l-n_2}}{(q)_{l-m_1-n_1} (\bar{q})_{l-m_2-n_2}} \times$$

$$q^{(m_2-m_1)(l-m_1-m_2)+(n_2-n_1)(l-n_1-n_2)+m_2-m_1+n_2-n_1}.$$

The program to compute $J_{N+1}(K)/J_N(K)$ is almost equal to those for the previous examples. The only different lines are the following.

```
...

ans = sum(ll=0, N-1, labs[ll+1]*\
 sum(m1=0, ll, labs[ll-m1+1]*labs[N-1-m1+1]*\
  sum(n1=0, ll-m1, labs[N-1-n1+1]*lm[ll-n1+1]*lm[N-1-ll+m1+n1+1]*\
   sum(m2=0, ll-m1, labs[ll-m2+1]*labs[N-1-m2+1]*\
    sum(n2=0, ll-m2, labs[N-1-n2+1]*l[ll-n2+1]*l[N-1-ll+m2+n2+1}*\
      Mod(x^(((m2-m1)*(ll-m1-m2)+(n2-n1)*(ll-n1-n2)+m2-m1+n2-n1)%N), \
x^N-1))))))

...

2*Pi*log(ans2*(N-1)^10/ans1/N^10)
```

The results are given in Table 5.

N	$2\pi \log \frac{J_{N+1}(K)}{J_N(K)}$
5	8.036805097240829695180371009
10	8.373856508425248006124939747
15	8.152791235806956158626064554
20	8.021952312877980724820244796
25	7.941218675423634478989298960
30	7.885684247868739884080382928
40	7.814415752862457696272810490
50	7.770664225432679874868903250
$2\pi \log \frac{J_{51}(K)}{J_{50}(N)} - \frac{3\pi}{50}$	7.582168666217292280561144647
$\mathrm{Vol}(K)$	7.5881802236416

TABLE 5

Graph. The points $\left(\dfrac{1}{N}, 2\pi \log \dfrac{J_{N+1}(K)}{J_N(K)}\right)$ are plotted as follows.

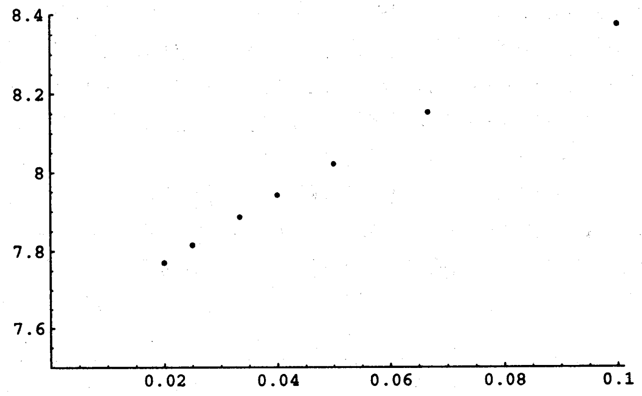


FIGURE 5. Plotting of the points $\left(\dfrac{1}{N}, 2\pi \log \dfrac{J_{N+1}(K)}{J_N(K)}\right)$ of the knot 8_9.

3.7. **Knot 8_{20}.** Let K be the knot 8_{20}. Then

$$J_N(K) = \sum_{\substack{j,l \le k \le i+l \le j+m \\ j \le i \\ 0 \le i,j,k,l,m \le N-1}} \frac{\{(\bar{q})_i (q)_k (\bar{q})_m\}^2}{\{(\bar{q})_j (q)_l\}\,(q)_{k-l}(\bar{q})_{i-k+l}(\bar{q})_{j+m-i-l}(q)_{i-j}(q)_{k-j}}\, q^{k+m+im+km-il}.$$

Program.

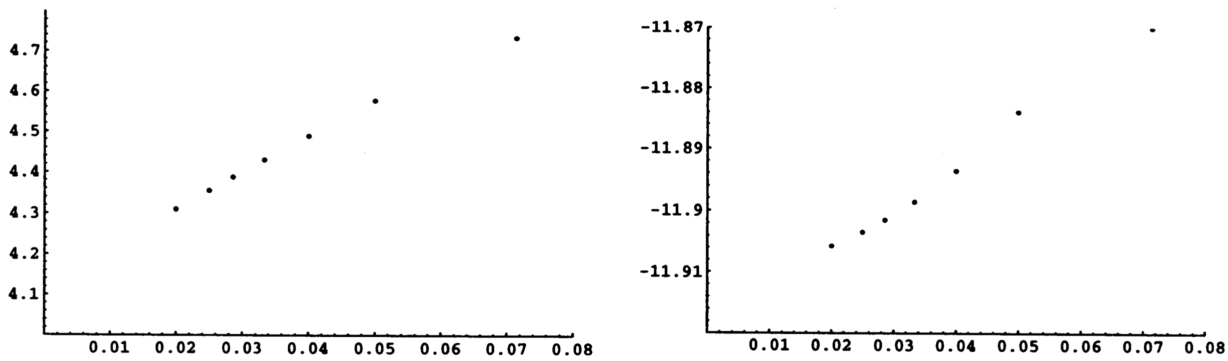
```
...

ans = sum(i=0, N-1, lm2[i+1]*\
 sum(j=0, i, l2[N-1-j+1]*lm[N-1-i+j+1]*\
  sum(ll=0, N-1, lm2[N-1-ll+1]*\
   sum(k=max(ll,j), min(N-1, i+ll), \
     l2[k+1]*lm[N-1-k+ll+1]*l[N-1-i+k-ll+1]*lm[N-1-k+j+1]*\
    sum(m=i-j+ll, N-1, lm2[m+1]*l[N-1-j-m+i+ll+1]*\
       Mod(x^((k+m+i*m+k*m-i*ll)%N), x^N-1)\
)))))

...

2*Pi*log(ans2*(N-1)^9/ans1/N^9)
```

Results.

N	$2\pi \log \dfrac{J_{N+1}(K)}{J_N(K)}$
5	4.993134830282317119922109736 - 8.534810138421228059039058370$\sqrt{-1}$
7	6.058772085097703463174557594 - 13.010024626707872888667162 57$\sqrt{-1}$
10	4.838146313755788700051905369 - 11.947294009269427216372130 50$\sqrt{-1}$
14	4.733597316958210595817845225 - 11.870057526206257918775406 83$\sqrt{-1}$
20	4.577298093617009639760204539 - 11.883968706715133447941561 34$\sqrt{-1}$
25	4.488646016707939440733681448 - 11.893625529242850780827340 17$\sqrt{-1}$
30	4.429868129481447562532108244 - 11.898595948889740902360890 72$\sqrt{-1}$
35	4.387402367809206482628251653 - 11.901539277734299543761455 56$\sqrt{-1}$
40	4.355311811237171523425351164 - 11.903471831929300815077739 09$\sqrt{-1}$
50	4.310046749251591944060510784 - 11.905764696229715866687614 26$\sqrt{-1}$
$2\pi \log \frac{J_{51}(K)}{J_{50}(N)} - \frac{3\pi}{50}$	4.121551190036204349752752181 - 11.905764696229715866687614 26$\sqrt{-1}$
Vol$(K)+$ $\sqrt{-1}$ CS(K)	4.1249032518077 -11.9099170709 $\sqrt{-1}$

TABLE 6. $\mathrm{CS}(K) = -2\pi^2 \,\mathrm{cs}(K) - \pi^2$.

Graphs.



real part

imaginary part

FIGURE 6. Plotting of the points $\left(\dfrac{1}{N}, 2\pi \log \dfrac{J_{N+1}(K)}{J_N(K)}\right)$ of the knot 8_{20}.

3.8. **Whitehead link.** The Whitehead link is the most simplest hyperbolic 2-component link. It is not a one-component knot, but complexified hyperbolic volume conjecture seems to hold for this link as follows.

Let K be the Whitehead link. Then

$$J_N(K) = \sum_{\substack{0 \le i,j,k \le N-1 \\ k \le i,j}} \frac{\{(\bar{q})_i (\bar{q})_j\}^2}{(q)_k{}^4 (\bar{q})_{i-k} (\bar{q})_{j-k}} q^{-(N-1)N/2}$$

Program.

```
...

ans = sum(k=0, N-1, lm4[N-1-k+1]*\
        sum(i=k, N-1, lm2[i+1]*l[N-1-i+k+1]*\
          sum(j=k, N-1, lm2[j+1]*l[N-1-j+k+1])))

...

2*Pi*log(ans2*(N-1)^6/ans1/N^6)
```

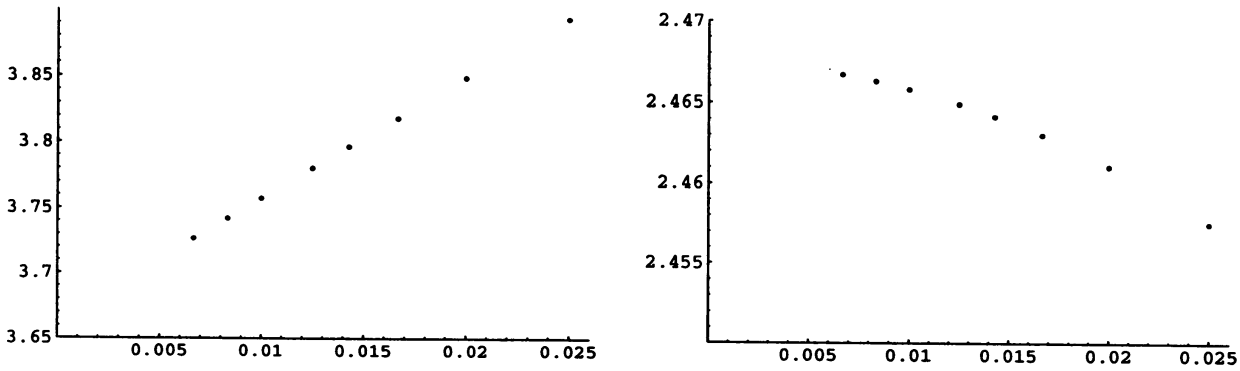
Here $q^{-(N-1)N/2}$ is omitted since it is equal to ± 1, which contributes to $\mathrm{CS}(K)$ by a multiple of $2\pi^2$.

Results.

N	$2\pi \log \dfrac{J_{N+1}(K)}{J_N(K)}$
40	$3.892920359101811097809525583 + 2.457483997330866045812504703\sqrt{-1}$
50	$3.848161466402914225154530180 + 2.461039474018016569869745301\sqrt{-1}$
60	$3.818029013349499312708236153 + 2.462976748675980254703390855\sqrt{-1}$
70	$3.796362501209537691078944556 + 2.464147191795881614582476451\sqrt{-1}$
80	$3.780034327560022195082015385 + 2.464907923404764622274395868\sqrt{-1}$
100	$3.757062258985477857247991239 + 2.465803785962819679236327339\sqrt{-1}$
120	$3.741674608179023673159144258 + 2.466291085896660260688606142\sqrt{-1}$
150	$3.726228649726558590507828429 + 2.466690204011030007962113880\sqrt{-1}$
$2\pi \log \frac{J_{151}(K)}{J_{150}(N)} - \frac{3\pi}{150}$	$3.663396796654762725738575561 + 2.466690204011030007962113880\sqrt{-1}$
$\mathrm{Vol}(K) + \sqrt{-1}\,\mathrm{CS}(K)$	$3.6638623767089 + 2.46740110027234\sqrt{-1}$

TABLE 7. $\mathrm{CS}(K) = 2\pi^2 \,\mathrm{cs}(K)$, where $cs(K) = 1/8$.

Graphs.



real part imaginary part

FIGURE 7. Plotting of the points $\left(\frac{1}{N}, 2\pi \log \frac{J_{N+1}(K)}{J_N(K)}\right)$ of the Whitehead link.

Fitting.

$$3.66386 + 2.46742\sqrt{-1} + \left(9.42575 - 0.00497353\sqrt{-1}\right)\frac{1}{N} - \left(10.5298 + 15.7048\sqrt{-1}\right)\frac{1}{N^2}$$

Remark. The volume conjecture (4) is not hold for all links, because $J_N(L) = 0$ if L is a split link $L = K_1 \sqcup K_2$. In this case,

$$|S^3 \setminus L| = |S^3 \setminus K_1| + |S^3 \setminus K_2|$$

and so

$$\lim_{N\to\infty} \exp\left(N\,|S^3 \setminus L|\right)$$

does not equal to 0 if K_1 and K_2 are both hyperbolic knots.

4. Conclusion

In the above computations, we see the behavior of $\dfrac{J_{N+1}(K)}{J_N(K)}$ to chech the formula (6). We also compare with Hikami's obserbation (7). Both conjectures (6) and (7) seem to be true for the examples given here. Moreover, the imaginary part of the coefficient of $\dfrac{1}{N}$ in the asymptotic expansion of $\dfrac{J_{N+1}(K)}{J_N(K)}$ seems to be 0 for these examples.

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